Radon transformation

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Here I derive the inverse of Radon transform [1].

The 2D Radon transform gives what one usually measures in a tomography experiment. Given a function $f(x,y)$ of which you want to get information, you may be able to measure its integration over a straight line with distance r to the origin point, whose normal vector has an angle θ with the x-axis. See Fig. 1. We denote this transformation by

$$[\mathcal{R}f](\theta,r) = \int_{-\infty}^{\infty} \mathrm{d}s\, f(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)\,. \tag{1}$$

We want to derive its inverse transform.

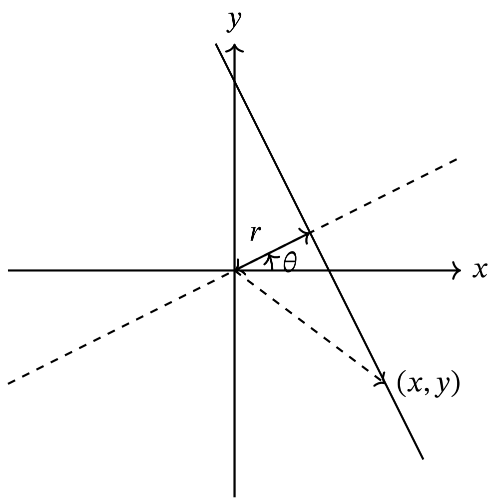

Figure 1: Illustration of Radon transformation

It is easy to see that, the 1D Fourier transform of $\mathcal{R}f$ with argument r is equivalent to the 2D Fourier transform of f, since for any (x,y) on the integration line, its inner product with (θ,k) equals rk. Explicitly, we have

$$[\mathcal{F}_r\mathcal{R}f](\theta,k) = \sqrt{2\pi}\left[\mathcal{F}_{(x,y)}f\right](k\cos\theta, k\sin\theta)\,, \tag{2}$$

where

$$[\mathcal{F}_x f](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\mathrm{e}^{-\mathrm{i}kx}, \quad \left[\mathcal{F}_{(x,y)}f\right](k_x,k_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\, f(x,y)\mathrm{e}^{-\mathrm{i}(k_x x + k_y y)}. \tag{3}$$

As a result,

$$\begin{aligned} f(x,y) &= \frac{1}{\sqrt{2\pi}}\left[\mathcal{F}^{-1}_{(x,y)}\mathcal{F}_r\mathcal{R}f\right](x,y) \\ &= \frac{1}{4\pi^2}\int_0^{\infty} k\,\mathrm{d}k\int_0^{2\pi}\mathrm{d}\theta\int_{-\infty}^{\infty}\mathrm{d}r\ \exp\left(\mathrm{i}k\left(x\cos\theta + y\sin\theta - r\right)\right)[\mathcal{R}f](\theta,r)\,. \end{aligned} \tag{4}$$

Define

$$K(x) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\xi \ |\xi|\, \mathrm{e}^{\mathrm{i}\xi x} = -\frac{\mathrm{P}}{x^2}, \tag{5}$$

and then we have

$$f(x,y) = \frac{1}{2\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{\infty} \mathrm{d}r\, K\left(x\cos\theta + y\sin\theta - r\right)\left[\mathcal{R}f\right](\theta, k)\,. \tag{6}$$

This is the inverse Radon transform.

References

[1] J. Radon, *Über die bestimmung von funktionen durch ihre integralwere längs gewisser mannigfaltigkeiten.*, *Verh. Sachs. Akad. Wiss. Leipzig, Math. Phys. Klass.* (1917) 262.