Topics on Group Theory

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Abstract

This note is about several topics on group theory. Completeness is not expected.
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Contents

1 Direct product and Semidirect product

1.1 Direct product

First define the direct product of two groups [1].

Definition 1.1 (Direct product of groups)**.** *Given groups G and H, the direct group $G \times H$ is defined as*

DP1 The underlying set is the Cartesian product $G \times H$. The elements are denoted as (g, h) where $g \in G$ and $h \in H$.

DP2 The operation is given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2) . \tag{1}$$

1.2 Semidirect product

A more complex and common structure is the semidirect product [2]. We can first prove that the following statements are equivalent.

Theorem 1.1. *Given a group G with identity element e, a subgroup H and a normal subgroup $N \lhd G$, the following statements are equivalent:*

SDP1 G is the product of subgroups, $G = NH$, and these subgroups have trivial intersection: $N \cap H = \{e\}$.

SDP2 For every $g \in G$, there are unique $n \in N$ and $h \in H$ such that $g = nh$.

SDP3 For every $g \in G$, there are unique $h \in H$ and $n \in N$ such that $g = hn$.

SDP4 The composition $\pi \circ i$ of the natural embedding $i : H \to G$ with the natural projection $\pi : G \to G/N$ is an isomorphism between H and the quotient group G/N.

SDP5 There exists a homomorphism $G \to H$ that is the identity on H and whose kernel is N.

Proof. SDP1 $\to$ SDP2: Since $G = NH$, there exist $n \in N$ and $h \in H$ such that $g = nh, \forall g \in G$. If there are $n' \in N, h' \in H$ such that $nh = n'h'$, we have $n'^{-1}n = h'h^{-1} = e$ and therefore $n = n'$ and $h = h'$ since $N \cap H = \{e\}$.

SDP2 $\to$ SDP3: Since N is normal, there exist $n' = h^{-1}nh \in N$ such that $nh = hh^{-1}nh = hn'$ for $n \in N$ and $h \in H$. If there exist $n'' \in N$ and $h'' \in H$ such that $nh = hn' = h''n''$, we have $nh = h''n''h''^{-1}h''$. Since the decomposition $g = hn$ is unique, $h'' = h$ and $n'' = h^{-1}nh = n'$, i.e., the decomposition $g = hn$ is also unique.

SDP3 $\to$ SDP4: Clearly that $\pi \circ i(h) = hN$. Since N is normal, the map is homomorphism. For any $g \in G$ we have $h \in H$ and $n \in N$ such that $gN = hnN = hN$, so the map is surjective. For $h, h' \in H$, $hN = h'N$ if and only if there exist $g \in G$, $n, n' \in N$ such that $hn = h'n' = g$. However, due to SDP3 $h = h'$, so the map is injective. Therefore the map $\pi \circ i$ is isomorphism.

SDP4 $\to$ SDP5: Define a map $\varphi : G \to H$ such that $\varphi(g) = h$ if and only if $g \in \pi \circ i(h)$ defined in SDP4. Since cosets of a subgroup is either the same or nonintersecting, this definition is reasonable. Clearly that φ is homomorphism and is the identity on H and has a kernal N directly due to SDP4.

SDP5 $\to$ SDP1: Suppose the homomorphism is φ. For an arbitrary $g \in G$, suppose that $\varphi(g) = h \in H$, then we have $\varphi\left(gh^{-1}\right) = e$ and therefore $gh^{-1} \in N$. That is, $G = NH$. For any $h \in H$, $h \in N$ if and only if $\varphi(h) = e$, that is, $h = e$. □

With the above theorem, we can define and immediately get several properties of the inner semidirect product of a group.

Definition 1.2 (Inner semidirect product)**.** *Given a group G with identity element e, a subgroup H and a normal subgroup $N \lhd G$, if any of the conditions among SDP1 to SDP5 holds, G is called the semidirect product of N and H, written*

$$G = N \rtimes H \text{ or } G = H \ltimes N. \tag{2}$$

Since every element in $G = N \rtimes H$ can be decomposited as nh with $n \in N$ and $h \in H$, consider the product of two elements $g_1 = n_1h_1$ and $g_2 = n_2h_2$,

$$g_1g_2 = n_1h_1n_2h_2 = n_1h_1n_2h_1^{-1}h_1h_2 = n_1\varphi_{h_1}(n_2)\,h_1h_2, \tag{3}$$

where $\varphi_h(n) = hnh^{-1}$. Due to SDP2, the decomposition is unique, so we can denote $g \in G$ as an ordered pair (n, h), with the product law

$$(n_1, h_1) \cdot (n_2, h_2) = \left(n_1\varphi_{h_1}(n_2)\,, h_1h_2\right), \tag{4}$$

where φ_h is defined above.

Motivated by the inner semidirect product, we can define the outer semidirect product in a similar way. Given two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$[1], define a product $\cdot$ on the Cartesian product of N and H,

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1, h_2). \tag{5}$$

We can verify whether it forms a group.

First, for $(n_1, h_1), (n_2, h_2), (n_3, h_3) \in N \times H$, we have the associativity

$$\begin{aligned}((n_1, h_1) \cdot (n_2, h_2)) \cdot (n_3, h_3) &= (n_1 \varphi(h_1)(n_2), h_1 h_2) \cdot (n_3, h_3) \\ &= (n_1 \varphi(h_1)(n_2) \varphi(h_1 h_2)(n_3), h_1 h_2 h_3) \\ &= (n_1 \varphi(h_1)(n_2 \varphi(h_2)(n_3)), h_1 h_2 h_3) \\ &= (n_1, h_1) \cdot ((n_2, h_2) \cdot (n_3, h_3)).\end{aligned} \tag{6}$$

Second, for every $(n, h) \in N \times H$,

$$(e, e) \cdot (n, h) = (n, h). \tag{7}$$

So (e, e) is the identity element.

Third, for every $(n, h) \in N \times H$, we have

$$\left(\varphi\left(h^{-1}\right)\left(n^{-1}\right), h^{-1}\right) \cdot (n, h) = (e, e). \tag{8}$$

Therefore we can define a group as follows.

Definition 1.3. *Given two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, we can construct a new group $N \rtimes_\varphi H$, called the outer semidirect product of N and H with respect to φ, defined as*

OSP1 The underlying set is the Cartesian product $N \times H$.

OSP2 The group operation is defined as

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1, h_2), \tag{9}$$

for $n_1, n_2 \in N$ and $h_1, h_2 \in H$.

To see the connection between the outer semidirect product and the inner semidirect product, consider the group $G = N \rtimes_\varphi H$. Define two subgroups (easy to verify) $\widetilde{N} = \{(n, e) \in G | n \in n\}$ and $\widetilde{H} = \{(e, h) \in G | h \in H\}$. For any $g = (n, h) \in G$ and $\widetilde{n} = (n', e) \in \widetilde{N}$,

$$g \widetilde{n} g^{-1} = (n, h) \cdot \left(n' \varphi\left(h^{-1}\right)\left(n^{-1}\right), h^{-1}\right) = \left(n \varphi(h)(n') n^{-1}, e\right) \in \widetilde{N}. \tag{10}$$

We have immediately that $\widetilde{N}$ is a normal subgroup and that we can define $\varphi_{\widetilde{h}}(\widetilde{n}) = \widetilde{h} \widetilde{n} \widetilde{h}^{-1}$ for $\widetilde{h} = (e, h) \in \widetilde{H}$ and $\widetilde{n} = (n, e) \in \widetilde{N}$ such that G is the inner semidirect product of $\widetilde{N}$ and $\widetilde{H}$ with the same group operation.

[1] The group of automorphisms on N. A simple example is that if $N \lhd G$, $n \to gng^{-1}$ with $n \in N$ and $g \in G$ is an automorphism.

1.3 Examples

All translations and rotations in $\mathbb{R}^3$ forms a group which is the semidirect product of translation group and rotation group, $\mathbb{R}^3 \rtimes_\varphi \mathrm{SO}(3)$, where $\varphi : \mathrm{SO}(3) \to \mathrm{Aut}\left(\mathbb{R}^3\right)$ is defined as

$$\varphi(R)\left(T(\boldsymbol{x})\right) = T\left(R\boldsymbol{x}\right) \tag{11}$$

with $T\left(\boldsymbol{x}\right) \in \mathbb{R}^3$ the translation by $\boldsymbol{x}$ and $R \in \mathrm{SO}(3)$ a rotation. The general element of the group is $(T(\boldsymbol{x}), R)$, with the product

$$\left(T\left(\boldsymbol{x_1}\right), R_1\right) \cdot \left(T\left(\boldsymbol{x_2}\right), R_2\right) = \left(T\left(\boldsymbol{x_1}\right) T\left(R_1 \boldsymbol{x_2}\right), R_1 R_2\right). \tag{12}$$

If we interprete $\left(T\left(\boldsymbol{x}\right), R\right)$ as rotation by R followed a translation by $\boldsymbol{x}$, the product above has the proper physical meaning. Of course we can view (T, R) as a product TR in the whole group, with the latter interpreted as the inner semidirect product of $\mathbb{R}^3$ and SO(3).

A widely use generalisation of the above example is the Poincaré group

$$\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3), \tag{13}$$

which forms the foundation of relativistic quantum field theory.

2 Basic facts of Lie groups

Here we discuss the Basic facts of Lie groups in two ways, the first of which is by the abstract, manifold language, and the second of which is by the coordinates and matrix elements, focusing on the representations.

2.1 Lie group and its Lie algebra

This section is based on the book [3].

Definition 2.1 (Lie group)**.** *A Lie group G is a differentiable manifold endowed with a group structure that the group operations $G \times G \to G$ and the map $G \to G$ defined by $g \mapsto g^{-1}$ are differentiable. If the dimension of the underlying manifold is r, we say that G is an r-parameter Lie group.*

In an r-parameter local Lie group we can define the multiplication of group elements explicitly. Consider a chart (U, ϕ) of the origin of $\mathbb{R}^r$ and the group structure of the Lie group G can be determined locally on U by a map $f : U \times U \to U$, fulfilling that

$$f(x, y) = z \text{ if and only if } \phi(x) \cdot \phi(y) = \phi(z), \forall x, y, z \in U. \tag{14}$$

The fact that G is a manifold implies that f is infinitely differentiable.

Like the concepts in the finite group, define the group action.

Definition 2.2 (Group action)**.** *Let M be a manifold. A local group of transformations acting on M is a local Lie group G, an open subset U with the property $\{e\} \times M \subset U \subset G \times M$, and a msp $\psi : U \to M$ satisfying the following conditions:*

1. *If $(g, P) \in U$, $(h, \psi(g, P)) \in U$, and $(hg, P) \in U$, then*

$$\psi\left(h, \psi\left(g, P\right)\right) = \psi\left(hg, P\right). \tag{15}$$

2. *For all $P \in M$, $\psi(e, P) = P$.*

3. *If $(g, P) \in U$, then $(g^{-1}, \psi\left(g, P\right)) \in U$ and $\psi\left(g^{-1}, \psi(g, P)\right) = P$.*

We can denote $\psi(g, P)$ by $g \cdot P$ or gP.

Taking the manifold to be G itself, we can define the left translation by $g \in G$ as a diffeomorphism $L_g : G \to G$ by

$$L_g(h) = gh \quad \forall h \in G. \tag{16}$$

A vector field[2] ξ on G is called **left-invariant** if for each $g \in G$, ξ is L_g-related to itself, i.e.

$$L_{g*}\left(\xi(h)\right) = \xi(gh) \quad \forall g, h \in G. \tag{17}$$

The set of left-invariant vector fields on G is denoted by $\mathfrak{g}$. We can define right invariant ones R_g similarly.

It is clear that $\mathfrak{g}$ is a real vector space. Define a map $\phi : \mathfrak{g} \to T_e(G)$ by $\phi(\xi) = \xi(e)$. It can be shown that ϕ is a linear isomorphism. Therefore $\dim \mathfrak{g} = \dim T_e(G) = \dim G$.

Define the **Lie brackets** of vector fields by

$$[X, Y] = X \circ Y - Y \circ X, \tag{18}$$

where for convenience we use X to denote the map, which is naturally induced by the vector field X on the manifold M, from the set of differentiable functions on the manifold $\mathcal{F}(M)$ to itself. Under such a binary operation the vector field becomes an algebra, called **Lie algebra**. Since for any differentiable manifold N, any differentiable map $F : M \to N$, and any differentiable function $f : N \to \mathbb{R}$, we have[3]

$$F_*\left[X_1, X_2\right] f = \left[F_* X_1, F_* X_2\right] f, \tag{19}$$

the vector field $\mathfrak{g}$ is a Lie algebra with respect to the Lie brackets.

Now we can define the Lie algebra of a Lie group.

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[2]Recall that a vector field is a map from the manifold to the tangent bundle.

[3]Note that there is something tricky in deriving the equation below. For a tangent vector X_p with $p \in M$ and a differentiable function $f : N \to \mathbb{R}$, we have $\left(F_* X_p\right) f = X_p\left(f \circ F\right)$. However, if we consider a vector field X as the map $\mathcal{F}(M) \to \mathcal{F}(M)$ it induced, then $\left(F_* X\right) f$ will be a function on N, while $X(f \circ F)$ is a function on M. Here the tangent map of $F : M \to N$ aquires a new interpretation as a map from $\mathrm{Map}\left(\mathcal{F}(M), \mathcal{F}(M)\right)$ to $\mathrm{Map}\left(\mathcal{F}(N), \mathcal{F}(N)\right)$. Actually, we have $X\left(f \circ F\right) = \left(\left(F_* X\right) f\right) \circ F$.

Definition 2.3 (Lie algebra of a Lie group). *The Lie algebra of the Lie group G is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. Sometimes we think of ξ as a vector in $T_e(G)$. In that case, we denote by X_ξ the left-invariant vector field whose value at the identity is ξ.*

It is straightforward that $X_\xi(g) = L_{g*}\xi$.

If two groups stand in some algebraic relation to one another, their Lie algebras will inherit such relations. Its easy to prove the theorem below.

Theorem 2.1. *If $\phi : G \to H$ is a Lie group homomorphism, then $\phi^* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Now let us illustrate how, as the physicists understand, the Lie algebra can be used to generate the group.

First we show that a one-parameter subgroup of Lie group G induce a left-invariant vector field, and inversely, a left-invariant vector field generates a unique one-parameter subgroup. As for the first part, a one-parameter subgroup is a homomorphism $\gamma : \mathbb{R} \to G$ such that $\gamma(t+s) = \gamma(t)\gamma(s)$ and $\gamma(0) = e$. For a left-invariant vector field X such that $X_e = \dot{\gamma}(0)$ and a function $f : G \to \mathbb{R}$, note that

$$\begin{aligned} X_{\gamma(t)}f = \left(L_{\gamma(t)*}\dot{\gamma}(0)\right) f = \dot{\gamma}(0)\left(f \circ L_{\gamma(t)}\right) = \frac{\mathrm{d}}{\mathrm{d}u} f\left(\gamma(t)\gamma(u)\right)\bigg|_{u=0} \\ = \frac{\mathrm{d}}{\mathrm{d}u} f\left(\gamma(t+u)\right)\bigg|_{u=0} = \frac{\mathrm{d}}{\mathrm{d}t} f\left(\gamma(t)\right) = \dot{\gamma}(t) f. \end{aligned} \tag{20}$$

Therefore X is everywhere tangent to the curve. For the second part, recall that for any vector field there exists a local flow that induces it. As a result, given a left-invariant vector field X, there exists a neighbourhood U of the identity $e \in G$ and such a local flow σ_t on it that

$$X_h f = \frac{\mathrm{d}f\left(\sigma_t(h)\right)}{\mathrm{d}t}\bigg|_{t=0}. \tag{21}$$

Setting $\gamma(t) = \sigma_t(e)$, the condition $\gamma(t+s) = \gamma(t)\gamma(s)$ requires that $\sigma_t \circ L_g = L_g \circ \sigma_t$ and as a special case $\sigma_t(g) = g\sigma_t(e)$, which can be shown by directly arguing that $L_g \circ \sigma_t \circ L_{g^{-1}}$ generates the same vector field X and hence the same to σ_t. The local one-parameter subgroup can be extent to all values of t by setting

$$\gamma(t) = \left(\gamma(t/n)\right)^n. \tag{22}$$

One can check that $\gamma(t)$ is precisely a one parameter subgroup that induce the vector field X.

With the discussion above, we can define the exponential map.

Definition 2.4 (Exponential map). *The exponential map* $\exp : \mathfrak{g} \to G$ *is defined by*

$$\exp(X) = \gamma(1), \tag{23}$$

where $\gamma : \mathbb{R} \to G$ is the one-parameter subgroup generated by the left-invariant vector field X.

The exponential map has the following properties matching our intuition. First, define another one-perameter subgroup $\alpha(t) = \gamma(st)$, we can show that

$$\exp(sX) = \gamma(s). \tag{24}$$

And therefore

$$\exp(sX)\exp(tX) = \gamma(s)\gamma(t) = \gamma(s+t) = \exp\left((s+t)X\right). \tag{25}$$

And for a function $f : G \to \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f\left(\exp(tX)\right) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(\gamma(t)\right) = X_{\gamma(t)} f = (Xf)\left(\exp(tX)\right). \tag{26}$$

Now we can consider a local flow $F : \mathbb{R} \times G \to G$ generated by a vector field ξ in the Lie algebra

$$F_t(g) = g\exp(t\xi). \tag{27}$$

It is strightforward that $\dot{F}(g) = \xi(g)$.

From the definition of the Lie group, let $\phi : H \to G$ be a Lie group homomorphism. For any one-parameter subgroup $\gamma : \mathbb{R} \to H$, $\phi \circ \gamma$ is a one-parameter subgroup of G. The tangent vectors of the two one-parameter subgroups are linked by ϕ_*. Then for all $\eta \in \mathfrak{h}$, we have $\phi\left(\exp_H \eta\right) = \exp_G\left(\phi_* \eta\right)$.

Like what is done in the finite group theory, we can introduce for a group G the **inner aotumorphism** $I_g \equiv R_g^{-1} \circ L_g$, with $g \in G$. The Lie algebra isomorphism induced by the inner automorphism is denoted by $\mathrm{Ad}_g \equiv R_{g*}^{-1} \circ L_{g*}$. It is called the **adjoint map** associated with g. Since the Lie algebra $\mathfrak{g}$ is a vector space, the adjoint map can be use to construct a representation of G.

Definition 2.5 (Adjoint representation)**.** *The adjoint representation of a Lie group G is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(g) = \mathrm{Ad}_g$.*

The adjoint representation has the property that $\exp\left(\mathrm{Ad}_g \xi\right) = I_g \exp\xi = g\exp\xi g^{-1}$.

Choose a basis ξ_i for a finite dimensional Lie algebra $\mathfrak{g}$. The Lie bracket of two basis vectors is the combination of the basis vectors,

$$[\xi_i, \xi_j] = c_{ij}^k \xi_k. \tag{28}$$

Note that if we choose arbitrarily several vector fields, the coefficients for the linear decomposition of the Lie bracket may depent on the manifold points $g \in G$,

$$\left([\xi_i, \xi_j] f\right)(g) = \left(c_{ij}^k(g)\xi_k f\right)(g), \tag{29}$$

where $f \in \mathcal{F}(G)$[4]. But here since we are discussing a Lie algebra, the coefficients become constant for any group element $g \in G$. This is called the **Lie's second theorem**, and the constants is called the **structure constants**.

The structure constants has the properties following the antisymmetry and the Jacobi identity of the Lie brackets,

$$\begin{aligned} c^{\kappa}_{\rho\sigma} + c^{\kappa}_{\sigma\rho} &= 0, \\ c^{\kappa}_{\rho\sigma} c^{\nu}_{\kappa\mu} + c^{\kappa}_{\sigma\mu} c^{\nu}_{\kappa\rho} + c^{\kappa}_{\mu\rho} c^{\nu}_{\kappa\sigma} &= 0. \end{aligned} \tag{30}$$

The identities (30) is called the **Lie's third theorem**.

2.2 Infinitesimal action

References

[1] Direct product of groups. https://en.wikipedia.org/wiki/Direct_product_of_groups. Accessed: 2021-7-23.

[2] Semidirect product. https://en.wikipedia.org/wiki/Semidirect_product. Accessed: 2021-7-23.

[3] Sadri Hassani. *Mathematical physics: a modern introduction to its foundations*. Springer Science & Business Media, 2013.

[4] In the book [3] the author wrote the Lie brackets of two vectors $\xi_i(g)$, which is not correct.