Note on Quantum Field Theory

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Abstract

This note is about quantum field theory for spin-zero particles, based on the first part of [1].

In this note we take $c = \hbar = k_B = 1$. Einstein summation convention is taken.

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Contents

1 Lorentz invariance

In this section, we first define the Lorentz transformation and Lorentz group. Then we consider the infinitesimal transformation and the generator of the Lorentz group. Translation is also discussed and hence the Poincaré group. We discuss the Lie algebra of such groups.

1.1 Lorentz transformations and Lorentz group

The interval between x^μ and the origin is

$$x^2 = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = \boldsymbol{x}^2 - t^2, \tag{1.1}$$

where

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{1.2}$$

A Lorentz transformation is a linear transformation from x^μ to $\overline{x}^\mu$,

$$\overline{x}^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.3}$$

which preserves the interval. Therefore the matrix $\Lambda^\mu{}_\nu$ must obey

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{1.4}$$

The set of all Lorentz transformations forms a group, where the inverse of a transformation is

$$\left(\Lambda^{-1}\right)^\rho{}_\nu = \Lambda_\nu{}^\rho, \tag{1.5}$$

for $\Lambda_\nu{}^\rho \Lambda^\nu{}_\sigma = g_{\nu\mu} g^{\rho\lambda} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma = g_{\lambda\sigma} g^{\rho\lambda} = \delta^\rho_\sigma$. The determinant of Lorentz matrices can be ± 1. The transformations with determinant $+1$ is called proper. Note that $\left(\Lambda^0{}_0\right)^2 - \Lambda^i{}_0 \Lambda^i{}_0 = 1$, therefore $|\Lambda^0{}_0| \geqslant 1$. The transformations with $\Lambda^0{}_0 \geqslant 1$ is called orthochronous. The Lorentz group is thus divided into four branches. The proper orthochronous subgroup can be generated by infinitesimal transformation which we will discuss later. Two discrete transformations are introduced to connect the different branches, which are parity transformation

$$\mathcal{P}^\mu{}_\nu = \left(\mathcal{P}^{-1}\right)^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.6}$$

and time-reversal transformation

$$\mathcal{T}^\mu_{\ \nu} = \left(\mathcal{T}^{-1}\right)^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{1.7}$$

1.2 Generators and Lie algebra of Lorentz and Poincaré group

The infinitesimal transformation is defined as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu. \tag{1.8}$$

From equation (1.4) we know that $\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$. Therefore there are six independent infinitesimal Lorentz transformations including three rotations ($\delta\omega_{ij} = -\epsilon_{ijk}\widehat{n}_k\delta\theta$) and three boosts ($\delta\omega_{i0} = \widehat{n}_i \delta\eta$). Considering unitary representation of Lorentz transformations, we have

$$U(1+\delta\omega) = I + \frac{\mathrm{i}}{2}\delta\omega_{\mu\nu}M^{\mu\nu}, \tag{1.9}$$

where $M^{\mu\nu} = -M^{\nu\mu}$ is a set of hermitian operators called the generators of the Lorentz group. Consider $\Lambda' = 1 + \delta\omega'$,

$$U\left(\Lambda^{-1}\Lambda'\Lambda\right) = I + \frac{\mathrm{i}}{2}\Lambda_{\mu\rho}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} = I + \frac{\mathrm{i}}{2}\delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}. \tag{1.10}$$

And

$$U\left(\Lambda^{-1}\Lambda'\Lambda\right) = U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = I + \frac{\mathrm{i}}{2}\delta\omega'_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda). \tag{1.11}$$

For $\delta\omega'$ is arbitrary, we have

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}. \tag{1.12}$$

This Lorentz transformation for operators is general. For example, consider the generator of translation P^{μ},

$$T(\delta x)x = x + \delta x, \quad U(T(\delta x)) = I - \mathrm{i}\delta x_{\mu}P^{\mu}, \tag{1.13}$$

we have similarly

$$U\left(\Lambda^{-1}T(\delta x)\Lambda\right) = U\left(T\left(\Lambda^{-1}\delta x\right)\right) = I - \mathrm{i}\delta x_{\mu}\Lambda^{\mu}{}_{\nu}P^{\nu}, \tag{1.14}$$

that is,

$$U(\Lambda)^{-1}P_{\mu}U(\Lambda) = \Lambda^{\mu}{}_{\nu}P^{\nu}. \tag{1.15}$$

Note that the translation together with the Lorentz transformation form the Poincaré group.

To specify the Lie algebra of the Lorentz and Poincaré group, take $\Lambda = 1 + \delta\omega$ in equation (1.12) and expand to the first order in $\delta\omega$

$$\begin{aligned} M^{\mu\nu} + \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} &= \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \\ &= U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) \\ &= M^{\mu\nu} - \frac{\mathrm{i}}{2}\delta\omega_{\rho\sigma}M^{\rho\sigma}M^{\mu\nu} + \frac{\mathrm{i}}{2}\delta\omega_{\rho\sigma}M^{\mu\nu}M^{\rho\sigma}, \end{aligned} \tag{1.16}$$

from which we get

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -2\mathrm{i}\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma}\right) \\ &= \mathrm{i}\left(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma). \end{aligned} \tag{1.17}$$

Similarly, we can get from equation (1.15) that

$$[P^{\mu}, M^{\rho\sigma}] = \mathrm{i}\left(g^{\mu\sigma}P^{\rho} - (\rho \leftrightarrow \sigma)\right). \tag{1.18}$$

Now define the angular momentum operator $\boldsymbol{J}$ as $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ and the boost operator $\boldsymbol{K}$ as $K_i = M^{i0}$. Then we have the commutators

$$
\begin{aligned}
[J_i, J_j] &= \mathrm{i}\epsilon_{ijk}J_k, \\
[J_i, K_j] &= \mathrm{i}\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -\mathrm{i}\epsilon_{ijk}J_k.
\end{aligned} \tag{1.19}
$$

Furthermore, note that $P_\mu = (H, \boldsymbol{P})$, we have

$$
\begin{aligned}
[J_i, H] &= 0, \\
[J_i, P_j] &= \mathrm{i}\epsilon_{ijk}P_k, \\
[K_i, H] &= \mathrm{i}P_i, \\
[K_i, P_j] &= \mathrm{i}\delta_{ij}H, \\
[P^\mu, P^\nu] &= 0.
\end{aligned} \tag{1.20}
$$

The commutators above form the Lie algebra of the Poincaré group.

2 Canonical quantization of real scalar fields

2.1 Lorentz transformation on a scalar field

We first clarify the meaning of a transformation on a field. We have three kinds of things here: the physical space M which is a d-dimensional manifold, the coordinate space $\mathbb{R}^d$ and the tensor bundle $T^{(r,s)}M = \bigcup_{p\in M} T_p^{(r,s)}(M)$ where $T_p^{(r,s)}(M)$ is the tensor space at $p \in M$. The field is a map $\phi : M \mapsto T^{(r,s)}M$ such that $\phi(p) \in T_p^{(r,s)}(M)$. Furthermore, we have different coordinates $x : M \mapsto \mathbb{R}^d$ and $\overline{x} : M \mapsto \mathbb{R}^d$ connected by a transformation $\Lambda : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $\overline{x}(p) = \Lambda \circ x(p)$ which is simply denoted as $\overline{x} = \Lambda x$. Also, we usually deal with coordinates in calculation and as a result we are interested in $\varphi : \mathbb{R}^d \mapsto T^{(r,s)}M$ such that $\varphi \circ x = \phi$. For simplicity of calculation, we usually denote $\phi(p) = \varphi(x(p))$ as $\varphi(x)$. If we change the coordinate by Λ, we want to find a new $\overline{\varphi}$ that

$$\overline{\varphi} \circ \overline{x} = \varphi \circ x = \phi, \tag{2.1}$$

which means that

$$\overline{\varphi} \circ x(p) = \varphi \circ \Lambda^{-1} \circ x(p). \tag{2.2}$$

In the more common (but less exact) notation, we have

$$\overline{\varphi}(x) = \varphi\left(\Lambda^{-1}x\right). \tag{2.3}$$

The value of the expression above is a tensor but not its components. If the tensor is a scalar, that is, $r = s = 0$, we directly get its component. If it is not the case, to get the right components of the tensor, we need one more step of transformation to the tensor components. The full expression in components is

$$\begin{aligned}\phi(p) &= (\overline{\varphi}\circ\overline{x}(p))^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}\,\overline{\partial}_{\mu_1}\otimes\cdots\otimes\overline{\partial}_{\mu_r}\otimes \mathrm{d}\overline{x}^{\nu_1}\otimes\cdots\otimes\mathrm{d}\overline{x}^{\nu_s}\\ &= (\varphi\circ x(p))^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_r}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_s}\\ &= (\varphi\circ x(p))^{\mu'_1\cdots\mu'_r}{}_{\nu'_1\cdots\nu'_s}\,\frac{\partial\overline{x}^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial\overline{x}^{\mu_r}}{\partial x^{\mu'_r}}\frac{\partial x^{\nu'_1}}{\partial\overline{x}^{\nu_1}}\cdots\frac{\partial x^{\nu'_s}}{\partial\overline{x}^{\nu_s}}\overline{\partial}_{\mu_1}\otimes\cdots\otimes\overline{\partial}_{\mu_r}\otimes\mathrm{d}\overline{x}^{\nu_1}\otimes\cdots\otimes\mathrm{d}\overline{x}^{\nu_s}.\end{aligned}\tag{2.4}$$

In the special case of a linear transformation Λ, we have

$$\overline{\varphi}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(x) = \Lambda^{\mu_1}{}_{\mu'_1}\cdots\Lambda^{\mu_r}{}_{\mu'_r}\Lambda_{\nu_1}{}^{\nu'_1}\cdots\Lambda_{\nu_s}{}^{\nu'_s}\varphi^{\mu'_1\cdots\mu'_r}{}_{\nu'_1\cdots\nu'_s}\left(\Lambda^{-1}x\right). \tag{2.5}$$

We also need to consider integrals. An integral of function of tensors over a manifold is without doubt unchanged when we transform the coordinates. However, we will see some integrals over manifolds, with the integrand being a function of the coordinates and/or components of tensors. We may wonder whether such integrals can form a tensor, for which we need to know how the integral transforms under a (Lorentz) transformation. Generally, such integral can be written as

$$I = \int \mathrm{d}^d x(p)\, f\left(x^\mu(p), \phi^\mu{}_\nu(p)\right) \equiv \int \mathrm{d}^d x\, f\left(x^\mu, \varphi^\mu{}_\nu(x)\right), \tag{2.6}$$

and we want the transformed integral to be

$$\overline{I} = \int \mathrm{d}^d\overline{x}(p)\, f\left(\overline{x}^\mu(p), (\overline{\varphi}\circ\overline{x})^\mu{}_\nu(p)\right). \tag{2.7}$$

In the case of the Lorentz transformation $\Lambda^\mu{}_\nu$, we have

$$\begin{aligned}\overline{I} &= \int \mathrm{d}^4x\, f\left(\Lambda^\mu{}_\nu x^\nu, \Lambda^\mu{}_{\mu'}\Lambda_\nu{}^{\nu'}\varphi^{\mu'}{}_{\nu'}(x)\right) \quad \text{or}\\ &= \int \mathrm{d}^4x\, f\left(x, \Lambda^\mu{}_{\mu'}\Lambda_\nu{}^{\nu'}\varphi^{\mu'}{}_{\nu'}\left(\Lambda^{-1}x\right)\right).\end{aligned}\tag{2.8}$$

As an example, the integral over space $\int \mathrm{d}^3x$ should be interpreted as $\int \mathrm{d}^4x\,\delta(x^0)$, and the transformed integral is $\int \mathrm{d}^4x\,\delta\left(\Lambda^0{}_\mu x^\mu\right)$ or $\int \mathrm{d}^4x\,\delta\left(x^0\right)$ depending on the form we take in equation (2.8). To further illustrate this formalism, we will derive the famous length contraction effect by this transformation of integral. Consider a $1+1$-dimensional space (t, x), the length of a line sector between $(0,0)$ and $(0,L)$ is

$$\int \mathrm{d}t\,\mathrm{d}x\,\delta(t)\theta(x)\theta(L-x) = L, \tag{2.9}$$

where L in $\theta(L-x)$ should be viewed as the spatial coordinate of a constant vector field $v^\mu(x) = (0, L)$. In a moving frame of velocity β, we have

$$\begin{aligned}L' &= \int \mathrm{d}t\,\mathrm{d}x\,\delta\left(\gamma t - \beta\gamma x\right)\theta\left(\gamma x - \beta\gamma t\right)\theta\left(\gamma L - \gamma x + \beta\gamma t\right) = \gamma L \quad \text{or}\\ &= \int \mathrm{d}t\,\mathrm{d}x\,\delta(t)\theta(x)\theta\left(\gamma L - x\right) = \gamma L.\end{aligned}\tag{2.10}$$

Now turn to the Lorentz transformation of a scalar field. First consider the effect of a translation on a scalar field. For simplicity, we denote $U(T)$ as T. We have that

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a). \tag{2.11}$$

And for Lorentz transformation, we have

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi\left(\Lambda^{-1}x\right). \tag{2.12}$$

I don't know exactly how the unitary operators perform in the equation above, may be a generalisation of the operators in Heisenberg picture?

Derivatives of φ that carry vector indices transforms in the usual way

$$U(\Lambda)^{-1}\partial^\mu\varphi(x)\,U(\Lambda) = \Lambda^\mu{}_\rho\overline{\partial}^\rho\varphi\left(\Lambda^{-1}x\right), \tag{2.13}$$

where $\overline{\partial}$ is the derivative with respect to $\overline{x} = \Lambda^{-1}x$. Consequently,

$$U(\Lambda)^{-1}\partial^2\varphi(x)\,U(\Lambda) = \overline{\partial}^2\varphi\left(\Lambda^{-1}x\right). \tag{2.14}$$

2.2 Lorentz invariant real scalar field

To construct a Lorentz invariant real scalar field, one need to have an action being a scalar,

$$S = \int \mathrm{d}t\,L = \int \mathrm{d}^4x\,\mathcal{L}, \tag{2.15}$$

where $\mathcal{L}$ is the Lagrangian density. Since $\mathrm{d}^4\overline{x} = |\det\Lambda|\mathrm{d}^4x = \mathrm{d}^4x$ is Lorentz invariant, $\mathcal{L}$ should also be a scalar. Take for $\mathcal{L}$

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \Omega_0, \tag{2.16}$$

where Ω_0 is an arbitrary constant. The equation of motion is the Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu - m^2\right)\varphi = 0. \tag{2.17}$$

The solution of the Klein-Gordon equation can be expressed as a plane wave expansion

$$\varphi(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{f(k)}\left(a(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + b(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}\right), \tag{2.18}$$

where $\omega = +\left(\boldsymbol{k}^2+m^2\right)^{1/2}$. The four-vector product of $k^\mu = (\omega,\boldsymbol{k})$ and x is $kx = k^\mu x_\mu = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$. Note that the condition $k^2 = k^\mu k_\mu = -m^2$ defines the mass shell. Then we can write the solution in a more compact form

$$\varphi(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{f(k)}\left(a(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}kx} + b(-\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}kx}\right). \tag{2.19}$$

In order that the field be real, we should require that $a^*(\boldsymbol{k}) = b(-\boldsymbol{k})$, therefore the solution is

$$\varphi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{f(k)} \left(a(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + a^*(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \right). \tag{2.20}$$

In order that the solution be Lorentz invariant, we now determine $f(k)$. Since ${\Lambda^0}_0 - {\Lambda^0}_i {\Lambda^0}_i = 1 > 0$ and $(k^0)^2 - k^i k^i = m^2 > 0$, k^0 will not change its sign under an orthochronous Lorentz transformation, and an integration measure that is invariant under such transformation is $\mathrm{d}^4 k \, \delta(k^2 + m^2) \, \theta(k^0)$. The physical meaning of taking this integration measurement is that we only care about k's on the mass shell and having a positive k^0. Integral over k^0 we have[1]

$$\int_{-\infty}^{+\infty} \mathrm{d}k^0 \, \delta\left(k^2 + m^2\right) \theta\left(k^0\right) = \frac{1}{2\omega}. \tag{2.22}$$

Therefore we can take $f(k) = (2\pi)^3 2\omega$ and define

$$\widetilde{\mathrm{d}k} = \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega}. \tag{2.23}$$

Finally we have

$$\varphi(x) = \int \widetilde{\mathrm{d}k} \left(a(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + a^*(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \right). \tag{2.24}$$

To get $a(\boldsymbol{k})$ in terms of $\varphi(x)$, consider that

$$\begin{aligned} \int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i}kx} \varphi(x) &= \frac{1}{2\omega} a(\boldsymbol{k}) + \frac{1}{2\omega} \mathrm{e}^{2\mathrm{i}\omega t} a^*(-\boldsymbol{k}), \\ \int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i}kx} \partial_0 \varphi(x) &= -\frac{\mathrm{i}}{2} a(\boldsymbol{k}) + \frac{\mathrm{i}}{2} \mathrm{e}^{2\mathrm{i}\omega t} a^*(-\boldsymbol{k}). \end{aligned} \tag{2.25}$$

Then we have

$$a(\boldsymbol{k}) = \mathrm{i} \int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \varphi(x), \tag{2.26}$$

where $f \overleftrightarrow{\partial_\mu} g \equiv f(\partial_\mu g) - (\partial_\mu f) g$. Note that $a(\boldsymbol{k})$ is time-independent.

2.3 Canonical quantization of real scalar field

We are going to do canonical quantization to the field theory. This scheme of quantization relys on a selection of Cauchy slices supporting the Hamiltonian form and thus breaks the

[1] Note that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}, \tag{2.21}$$

where $g(x)$ is a smooth function of x with simple zeros as $x = x_i$.

Lorentz invariance, but if you calculate the physical observables, the Lorentz invariance is recovered.

The Hamiltonian form of the field is given by

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x), \tag{2.27}$$

and

$$\mathcal{H} = \Pi\dot{\varphi} - \mathcal{L} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \Omega_0. \tag{2.28}$$

The Hamiltonian density $\mathcal{H}$ is not Lorentz invariant. Write H in terms of $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$,

$$\begin{aligned}
H &= -\Omega_0 V + \frac{1}{2}\int \widetilde{\mathrm{d}k}\,\widetilde{\mathrm{d}k'}\,\mathrm{d}^3x\,\Big(\big(-\mathrm{i}\omega a(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}kx} + \text{h.c.}\big)\big(-\mathrm{i}\omega' a(\boldsymbol{k'})\,\mathrm{e}^{\mathrm{i}k'x} + \text{h.c.}\big)\\
&\quad + \big(\mathrm{i}\boldsymbol{k}a(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}kx} + \text{h.c.}\big)\cdot\big(\mathrm{i}\boldsymbol{k'}a(\boldsymbol{k'})\,\mathrm{e}^{\mathrm{i}k'x} + \text{h.c.}\big)\\
&\quad + m^2\big(a(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}kx} + \text{h.c.}\big)\big(a(\boldsymbol{k'})\,\mathrm{e}^{\mathrm{i}k'x} + \text{h.c.}\big)\Big)\\
&= -\Omega_0 V + \frac{1}{2}(2\pi)^3\int \widetilde{\mathrm{d}k}\,\widetilde{\mathrm{d}k'}\\
&\quad \times\Big(\delta^3(\boldsymbol{k}-\boldsymbol{k'})\big(\omega\omega' + \boldsymbol{k}\cdot\boldsymbol{k'} + m^2\big)\big(a(\boldsymbol{k})\,a^*(\boldsymbol{k'})\,\mathrm{e}^{-\mathrm{i}(\omega-\omega')t} + \text{h.c.}\big)\\
&\quad + \delta^3(\boldsymbol{k}+\boldsymbol{k'})\big(-\omega\omega' - \boldsymbol{k}\cdot\boldsymbol{k'} + m^2\big)\big(a(\boldsymbol{k})\,a(\boldsymbol{k'})\,\mathrm{e}^{-\mathrm{i}(\omega+\omega')t} + \text{h.c.}\big)\Big)\\
&= -\Omega_0 V + \frac{1}{2}\int \widetilde{\mathrm{d}k}\,\omega\,\big(a^*(\boldsymbol{k})\,a(\boldsymbol{k}) + a(\boldsymbol{k})\,a^*(\boldsymbol{k})\big).
\end{aligned} \tag{2.29}$$

Note that this quantity is still not Lorentz invariant.

In the Heisenberg picture, the canonical quantization can be done by introducing the following commutators at equal times

$$\begin{aligned}
[\varphi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{x'})] &= 0,\\
[\Pi(t,\boldsymbol{x}), \Pi(t,\boldsymbol{x'})] &= 0,\\
[\varphi(t,\boldsymbol{x}), \Pi(t,\boldsymbol{x'})] &= \mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{x'}).
\end{aligned} \tag{2.30}$$

Replace a^* by $a^\dagger$, we have

$$\begin{aligned}
\big[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})\big] &= -\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\mathrm{e}^{-\mathrm{i}(kx-k'x')}\,[\Pi(x) + \mathrm{i}\omega\varphi(x), \Pi(x') - \mathrm{i}\omega\varphi(x')]\\
&= -\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\mathrm{e}^{-\mathrm{i}(kx-k'x')}\,2\mathrm{i}\omega\mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{x'})\\
&= (2\pi)^3\,2\omega\delta(\boldsymbol{k}-\boldsymbol{k'}),
\end{aligned} \tag{2.31}$$

and similarly

$$\begin{aligned}
[a(\boldsymbol{k}), a(\boldsymbol{k'})] &= 0\\
\big[a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})\big] &= 0.
\end{aligned} \tag{2.32}$$

Substitute equations (2.31) and (2.32) into the Hamiltonian, we have

$$H = \int \widetilde{\mathrm{d}k}\, \omega a^{\dagger}(\boldsymbol{k})\, a(\boldsymbol{k}) + (\mathcal{E}_0 - \Omega_0)\, V, \tag{2.33}$$

where

$$\mathcal{E}_0 = \frac{1}{2}\int \mathrm{d}^3 k\, \omega \tag{2.34}$$

is the total zero-point energy that divergents. This divergence is called an ultraviolet divergence, which we can impose an ultraviolet cutoff $\Lambda \gg m$ or simply take $\Omega_0 = \mathcal{E}_0$ now.

The theory above is a field theory for bosons. If we replace the commutators by anticommutators, we expect to get a field theory for fermions but actually the Hamiltonian becomes trivial. If we go further we will find that for integer spins we can only take commutators to quantize the field theory and thus we have bosons, while for half integer spins we can only take anti-commutators and thus we have fermions. This is the spin-statistics theorem.

2.4 Translation operators for real scalar field

According to our discussion about the Noether's theorem, the conserved charge as well as the generators for translational symmetry is

$$H = \int \mathrm{d}^3 x\, \left(\Pi(x)\, \varphi(x) - \mathcal{L}\right) \tag{2.35}$$

for temporal translation, which is the Hamiltonian, and

$$\boldsymbol{P} = -\int \mathrm{d}^3 x\, \Pi(x)\, \nabla\varphi(x) \tag{2.36}$$

for spatial translation, which is the momentum. We can express $\boldsymbol{P}$ in terms of $a(\boldsymbol{k})$ and its Hermitian conjugates as what we have done for Hamiltonian,

$$\boldsymbol{P} = \int \widetilde{\mathrm{d}k}\, \boldsymbol{k} a^{\dagger}(\boldsymbol{k})\, a(\boldsymbol{k})\,. \tag{2.37}$$

3 The scattering amplitude

3.1 The LSZ reduction formula

In the following sections we will discuss the scattering amplitude which is closely related to experimentally measurable quantities.

First we should consider how to construct appropriate initial and final states for a scattering process. In the free theory, define a plane wave

$$|k\rangle = a^{\dagger}(\boldsymbol{k})\, |0\rangle. \tag{3.1}$$

With the vacuum state having the unit norm $\langle 0|0\rangle = 1$ we have

$$\langle k|k'\rangle = (2\pi)^3 \, 2\omega\delta^3 \left(\boldsymbol{k} - \boldsymbol{k'}\right), \tag{3.2}$$

where $\omega = \left(\boldsymbol{k}^2 + m^2\right)^{1/2}$. A 'particle' with momentum $\boldsymbol{k}_1$ and located near the origin in the free theory is a wave packet,

$$a_1^\dagger \equiv \int \mathrm{d}^3k \, f_1\left(\boldsymbol{k}\right) a^\dagger\left(\boldsymbol{k}\right), \tag{3.3}$$

where

$$f_1\left(\boldsymbol{k}\right) \propto \exp\left(-\left(\boldsymbol{k} - \boldsymbol{k'}\right)^2 / 4\sigma^2\right). \tag{3.4}$$

In the interaction theory, we guess that the initial state is

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger\left(t\right) a_2^\dagger\left(t\right)|0\rangle, \tag{3.5}$$

and the final state

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger\left(t\right) a_{2'}^\dagger\left(t\right)|0\rangle. \tag{3.6}$$

Note that in an interaction theory the $a_i^\dagger$'s are no longer time independent. The time evolution of $a_i^\dagger\left(t\right)$ can be derived as

$$\begin{aligned} a_i^\dagger\left(+\infty\right) - a_i^\dagger\left(-\infty\right) &= \int_{-\infty}^{+\infty} \mathrm{d}t \, \partial_0 a_i^\dagger\left(t\right) \\ &= -\mathrm{i}\int \mathrm{d}^3k \, f_i\left(\boldsymbol{k}\right) \int \mathrm{d}^4x \, \partial_0\left(\mathrm{e}^{\mathrm{i}kx}\overleftrightarrow{\partial_0}\varphi\left(x\right)\right) \\ &= -\mathrm{i}\int \mathrm{d}^3k \, f_i\left(\boldsymbol{k}\right) \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}kx}\left(-\partial^2 + m^2\right)\varphi\left(x\right). \end{aligned} \tag{3.7}$$

Therefore the scattreing amplitude

$$\langle f|i\rangle = \left\langle 0 \left| \mathrm{T}\left\{a_{1'}\left(+\infty\right) a_{2'}\left(+\infty\right) a_1^\dagger\left(-\infty\right) a_2^\dagger\left(-\infty\right)\right\}\right| 0\right\rangle, \tag{3.8}$$

can be calculated by introducing a time order T and substituting the equation (3.7) and taking $f_i\left(\boldsymbol{k}\right)$ as a delta function. The substituting replaces $a_i\left(+\infty\right)$ and $a_i^\dagger\left(-\infty\right)$ with $a_i^\dagger\left(+\infty\right)$ and $a_i\left(-\infty\right)$ which annihilates the vacuum state and only the product of the integrals in equaiton (3.7) survives. For a general case with n incoming particles and n' outcoming particles, we have the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula,

$$\begin{aligned} \langle f|i\rangle = \mathrm{i}^{n+n'} \int \mathrm{d}^4x_1 \, \mathrm{e}^{\mathrm{i}k_1x_1}\left(-\partial_1^2 + m^2\right)\cdots \mathrm{d}^4x_1' \, \mathrm{e}^{-\mathrm{i}k'x'}\left(-\partial_{1'}^2 + m^2\right)\cdots \\ \times \langle 0\left|\mathrm{T}\left\{\varphi\left(x_1\right)\cdots\varphi\left(x_1'\right)\cdots\right\}\right|0\rangle. \end{aligned} \tag{3.9}$$

The significance of this formula is that it connects the scattering matrix with the correlation functions.

3.2 Conditions on the field

This section is to be reviewed.

We need more conditions to ensure that $a_1^\dagger(\pm\infty)$ create a normalized single particle state.

First consider

$$\langle 0|\varphi(x)|0\rangle = \langle 0|\mathrm{e}^{-\mathrm{i}Px}\varphi(0)\,\mathrm{e}^{\mathrm{i}Px}|0\rangle = \langle 0|\varphi(0)|0\rangle. \tag{3.10}$$

We require this quantity to be zero in order that $a_1^\dagger(\pm\infty)$ does not create a superposition of the ground state and the single particle state.

Consider then

$$\langle p|\varphi(x)|0\rangle = \mathrm{e}^{-\mathrm{i}px}\langle p|\varphi(0)|0\rangle, \tag{3.11}$$

where $|p\rangle$ a normalized one particle state with four momentum p. The quantity $\langle p|\varphi(0)|0\rangle$ is required to be one in order that $a_1^\dagger(\pm\infty)$ creates the properly normalized state in the interaction theory as in the free theory.

To achieve the two requirements above, we need to rescale fields and some parameters in the Lagrangian.

4 Path integral

We have established a connection with scattering matrix and the correlation functions. Now we introduce a systematic approach, the path integral to calculate the correlation function.

4.1 Path integral in single particle quantum mechanics

In single particle quantum mechanics, it is well known that the inner product between two Heisenberg position operators is

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\,\mathcal{D}p\,\exp(\mathrm{i}S), \tag{4.1}$$

where

$$S = \int_{t'}^{t''} \mathrm{d}t\,\left(p(t)\dot{q}(t) - H(p(t), q(t))\right), \tag{4.2}$$

and integrating over $\mathcal{D}q$ is integrating over $\mathrm{d}q(t_i)$ with t_i's dividing the time interval, and $\mathcal{D}p$ is similar. The normalization constant is also absorbed in $\mathcal{D}q\,\mathcal{D}p$. If $H(p,q)$ is no more than quadratic in the momenta, then we can integrate p out and get

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\,\exp(\mathrm{i}S), \tag{4.3}$$

where

$$S = \int_{t'}^{t''} \mathrm{d}t\, L\left(q(t), \dot{q}(t)\right). \tag{4.4}$$

Given two functions $f(t)$ and $h(t)$, define

$$\langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\, \mathcal{D}q\, \exp\left(\mathrm{i} \int_{t'}^{t''} \mathrm{d}t\, \left(p\dot{q} - H + fq + hp\right) \right). \tag{4.5}$$

With the help of functional derivatives, we can calculate the time ordered correlator of position and momentum operators,

$$\langle q'', t'' | \mathrm{T}\left\{ Q\left(t_1\right) \cdots P\left(t_n\right) \cdots \right\} | q', t' \rangle = \frac{\delta}{\mathrm{i}\delta f\left(t_1\right)} \cdots \frac{\delta}{\mathrm{i}\delta h\left(t_n\right)} \cdots \langle q'', t'' | q', t' \rangle_{f,h} \Big|_{f=h=0}. \tag{4.6}$$

Now insert the ground states as the initial and final states and take $t' \to -\infty$ and $t'' \to +\infty$. Replace H with $(1 - \mathrm{i}\epsilon)\, H$ and denote the ground state as $|0\rangle$, we have

$$|q', t'\rangle = \mathrm{e}^{\mathrm{i}(1-\mathrm{i}\epsilon)Ht'} |q'\rangle = \sum_n \mathrm{e}^{\mathrm{i}(1-\mathrm{i}\epsilon)Ht'} |n\rangle\langle n|q'\rangle = \psi_0^*\left(q'\right)|0\rangle, \tag{4.7}$$

and

$$\langle q'', t''| = \langle 0|\psi_0\left(q''\right). \tag{4.8}$$

Therefore up to a normalization factor we have

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\, \mathcal{D}q\, \exp\left(\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t\, \left(p\dot{q} - (1 - \mathrm{i}\epsilon)\, H + fq + hp\right) \right). \tag{4.9}$$

For a perturbative problem with Hamiltonian $H = H_0 + H_1$, the path integral can be written as

$$\begin{aligned} \langle 0|0\rangle_{f,h} &= \int \mathcal{D}p\, \mathcal{D}q\, \exp\left(\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t\, \left(p\dot{q} - H_0\left(p, q\right) - H_1\left(p, q\right) + fq + hp\right) \right) \\ &= \exp\left(-\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t\, H_1\left(\frac{\delta}{\mathrm{i}\delta h(t)}, \frac{\delta}{\mathrm{i}\delta f(t)} \right) \right) \\ &\quad \times \int \mathcal{D}p\, \mathcal{D}q\, \exp\left(\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t\, \left(p\dot{q} - H_0\left(p, q\right) + fq + hp\right) \right). \end{aligned} \tag{4.10}$$

And for the case that $L_1 \equiv H_1$ depends only on q and that the path integral can be performed only over q, the expression can be simplified as

$$\langle 0|0\rangle_{f} = \exp\left(\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t\, L_1\left(\frac{\delta}{\mathrm{i}\delta f(t)} \right) \right) \int \mathcal{D}q\, \exp\left(\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t\, \left(L_0\left(\dot{q}, q\right) + fq\right) \right). \tag{4.11}$$

The $(1 - \mathrm{i}\epsilon)$ factor is suppressed in the equations above.

4.2 Path integral for harmonic oscillator

Consider a harmonic oscillator with hamiltonian

$$H(P,Q) = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2. \tag{4.12}$$

Take $m = 1$ and write the path integral

$$\langle 0|0\rangle_f = \int \mathcal{D}q \, \exp\left(\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}t \, \left(\frac{1+\mathrm{i}\epsilon}{2}\dot{q}^2 - \frac{(1-\mathrm{i}\epsilon)\,\omega^2}{2}q^2 + fq\right)\right). \tag{4.13}$$

Introducing the Fourier-transformed variables,

$$\widetilde{q}(E) = \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}Et} q(t), \quad q(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{2\pi}\, \mathrm{e}^{-\mathrm{i}Et}\widetilde{q}(E). \tag{4.14}$$

The action in the path integral can be rewritten as

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{2\pi} \, \left(\left(E^2 - \omega^2 + \mathrm{i}\left(E^2+\omega^2\right)\epsilon\right)\widetilde{q}(E)\widetilde{q}(-E) + \widetilde{f}(E)\widetilde{q}(-E) + \widetilde{f}(-E)\widetilde{q}(E)\right). \tag{4.15}$$

We can absorb the $(E^2 + \omega^2)$ factor into ϵ and change the integral variable to

$$\widetilde{x}(E) = \widetilde{q}(E) + \frac{\widetilde{f}(E)}{E^2 - \omega^2 + \mathrm{i}\epsilon}. \tag{4.16}$$

The action should then be

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{2\pi} \left(\widetilde{x}(E)\left(E^2-\omega^2+\mathrm{i}\epsilon\right)\widetilde{x}(-E) - \frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2-\omega^2+\mathrm{i}\epsilon}\right), \tag{4.17}$$

and the path integral

$$\begin{aligned}\langle 0|0\rangle_f = \exp&\left(\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}E}{2\pi}\,\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-\mathrm{i}\epsilon}\right)\\ &\times\int \mathcal{D}\widetilde{x}\,\exp\left(\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}E}{2\pi}\widetilde{x}(E)\left(E^2-\omega^2+\mathrm{i}\epsilon\right)\widetilde{x}(-E)\right).\end{aligned} \tag{4.18}$$

Note that $\langle 0|0\rangle_f\Big|_{f=0} = 1$, so we have

$$\langle 0|0\rangle_f = \exp\left(\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}E}{2\pi}\,\frac{\widetilde{f}(E)\widetilde{f}(-E)}{-E^2+\omega^2-\mathrm{i}\epsilon}\right), \tag{4.19}$$

or in terms of time-domain variables

$$\langle 0|0\rangle_f = \exp\left(\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G\left(t-t'\right)f\left(t'\right)\right), \tag{4.20}$$

where

$$G\left(t-t'\right) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}E}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}E(t-t')}}{-E^2+\omega^2-\mathrm{i}\epsilon} = \frac{\mathrm{i}}{2\omega}\exp\left(-\mathrm{i}\omega\left|t-t'\right|\right). \tag{4.21}$$

The function G is called the Green's function for it satisfies

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G\left(t-t'\right) = \delta\left(t-t'\right). \tag{4.22}$$

Now we can use equation (4.6) to calculate time-ordered products of position variables.

$$\langle 0\left|\mathrm{T}\left\{Q\left(t_1\right)\cdots Q\left(t_{2n}\right)\right\}\right|0\rangle = \frac{1}{\mathrm{i}^n}\sum_{\text{pairings}} G\left(t_{i_1}-t_{i_2}\right)\cdots G\left(t_{i_{2n-1}}-t_{i_{2n}}\right). \tag{4.23}$$

4.3 Path integral for free-field theory

Now consider the free-field theory with Hamiltonian density

$$\mathcal{H}_0 = \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{2}m^2\varphi^2. \tag{4.24}$$

Similar to the harmonic oscllator, we have the correspondence relation

$$\begin{aligned} q(t) &\to \varphi\left(t,\boldsymbol{x}\right) \quad \text{classical field},\\ Q(t) &\to \varphi\left(t,\boldsymbol{x}\right) \quad \text{operator field},\\ f(t) &\to J\left(t,\boldsymbol{x}\right) \quad \text{classical source}. \end{aligned} \tag{4.25}$$

Multiplying $\mathcal{H}$ by $(1-\mathrm{i}\epsilon)$ is equivalent to replacing m^2 with $m^2-\mathrm{i}\epsilon$, and we will not write it explicitly in the following discussion. The path integral for the free-field theory is now

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi\,\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\left(\mathcal{L}_0 + J\varphi\right)\right), \tag{4.26}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \tag{4.27}$$

is the Lagrangian density and

$$\mathcal{D}\varphi \propto \prod_x \mathrm{d}\varphi(x) \tag{4.28}$$

is the functional measure. Note that although the Hamiltonian density is not Lorentz invariant, the path integral is. We will further clarify how Lorentz invariance is broken or preserved in the two formalisms with the help of topological operators in later sections.

Similar to the case of harmonic oscillator, we can introduce the Fourier transformation of the fields,

$$\widetilde{\varphi}(k) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx}\varphi(x)\,, \quad \varphi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}kx}\widetilde{\varphi}(k)\,. \tag{4.29}$$

The action $S_0 = \int \mathrm{d}^4x\,(\mathcal{L}_0 + J\varphi)$ can be written as

$$S_0 = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(-\widetilde{\varphi}(k)\left(k^2+m^2\right)\widetilde{\varphi}(-k) + \widetilde{J}(k)\,\widetilde{\varphi}(-k) + \widetilde{J}(-k)\,\widetilde{\varphi}(k)\right). \tag{4.30}$$

Changing the variables to

$$\widetilde{\chi}(k) = \widetilde{\varphi}(k) - \frac{\widetilde{J}(k)}{k^2+m^2} \tag{4.31}$$

makes the action to be

$$S_0 = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{\widetilde{J}(k)\,\widetilde{J}(-k)}{k^2+m^2} - \widetilde{\chi}(k)\left(k^2+m^2\right)\widetilde{\chi}(-k)\right). \tag{4.32}$$

The path integral is now

$$\begin{aligned} Z_0(J) &= \exp\left(\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\widetilde{J}(k)\,\widetilde{J}(-k)}{k^2+m^2-\mathrm{i}\epsilon}\right) \\ &= \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,J(x)\,\Delta(x-x')\,J(x')\right), \end{aligned} \tag{4.33}$$

where the Feynman propagator

$$\begin{aligned} \Delta(x-x') &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2+m^2-\mathrm{i}\epsilon} \\ &= \mathrm{i}\theta(t-t')\int \widetilde{\mathrm{d}k}\,\mathrm{e}^{\mathrm{i}k(x-x')} + \mathrm{i}\theta(t'-t)\int \widetilde{\mathrm{d}k}\,\mathrm{e}^{-\mathrm{i}k(x-x')}, \end{aligned} \tag{4.34}$$

satisfies the differential equation

$$\left(-\partial_x^2 + m^2\right)\Delta(x-x') = \delta^4(x-x')\,. \tag{4.35}$$

To calculate the time ordered product of fields we can take functional derivatives with respect to J,

$$\langle 0\,|\mathrm{T}\{\varphi(x_1)\cdots\}|\,0\rangle = \frac{\delta}{\mathrm{i}\delta J(x_1)}\cdots Z_0(J)\Big|_{J=0}\,. \tag{4.36}$$

And therefore we have the Wick's theorem

$$\langle 0\,|\mathrm{T}\{\varphi(x_1)\cdots\varphi(x_{2n})\}|\,0\rangle = \frac{1}{\mathrm{i}^n}\sum_{\text{pairings}}\Delta(x_{i_1}-x_{i_2})\cdots\Delta\left(x_{i_{2n-1}}-x_{i_{2n}}\right). \tag{4.37}$$

which express the correlators with the Feynman propagators.

4.4 Path integral for interacting field theory

Let us consider a $\varphi - 3$ interacting field theory defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 + \frac{1}{6} Z_g g \varphi^3 + Y \varphi. \tag{4.38}$$

We fix the parameter m and g which is of physical meaning, and require that

$$\langle 0 | \varphi(x) | 0 \rangle = 0 \quad \text{and} \quad \langle k | \varphi(x) | 0 \rangle = \mathrm{e}^{-\mathrm{i}kx}. \tag{4.39}$$

Here $|0\rangle$ is the normalized ground state and $|k\rangle$ is a single particle state with four momentum k on the shell and normalized via

$$\langle k' | k \rangle = (2\pi)^3 2k^0 \delta^3 (\boldsymbol{k'} - \boldsymbol{k}). \tag{4.40}$$

Now we have four conditions to determine four parameters. We will take this example to illustrate the schemes for path integral for interacting theory although this model does not have a ground state.

Divide the Lagrangian density as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0, \tag{4.41}$$

where

$$\mathcal{L}_0 = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2, \tag{4.42}$$

and

$$\mathcal{L}_1 = \frac{1}{6} Z_g g \varphi + \mathcal{L}_{\mathrm{ct}}, \tag{4.43}$$

with

$$\mathcal{L}_{\mathrm{ct}} = -\frac{1}{2} (Z_\varphi - 1) \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} (Z_m - 1) m^2 \varphi^2 + Y \varphi. \tag{4.44}$$

The $\mathcal{L}_{\mathrm{ct}}$ is called the counterterm Lagrangian. We expect that as $g \to 0$, $Y \to 0$ and $Z_i \to 1$. Assume that $Y = O(g)$ and $Z_i = 1 + O(g^2)$. The path integral is then

$$Z(J) \propto \exp\left(\mathrm{i} \int \mathrm{d}^4 x \, \mathcal{L}_1 \left(\frac{\delta}{\mathrm{i}\delta J(x)} \right) \right) Z_0(J), \tag{4.45}$$

with Z_0 being the free-field path integral

$$Z_0(J) = \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^4 x \, \mathrm{d}^4 x' \, J(x) \Delta(x - x') J(x') \right). \tag{4.46}$$

In equation (4.45) we use proportional to rather than equald to for we will care less about the normalization factor and take $Z(0)$ to be one in the end.

Let us begin by ignoring the counterterms and define

$$Z_1(J) \propto \exp\left(\frac{\mathrm{i}}{6} Z_g g \int \mathrm{d}^4 x \left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\right) Z_0(J), \tag{4.47}$$

where the normalization condition is that $Z_1(0) = 1$. Make a dual Taylor expansion in powers of g and J,

$$\begin{aligned} Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!} &\left(\frac{\mathrm{i}Z_g g}{6} \int \mathrm{d}^4 x \left(\frac{\delta}{\mathrm{i}\delta J(x)}\right)^3\right)^V \\ &\times \sum_{P=0}^{\infty} \frac{1}{P!} \left(\frac{\mathrm{i}}{2} \int \mathrm{d}^4 y\, \mathrm{d}^4 z\, J(y)\, \Delta(y-z)\, J(z)\right)^P. \end{aligned} \tag{4.48}$$

We use Feynman diagrams to express the result. For a term with particular values of V and P, where V stands for vertex and P for propagator, the number of surviving sources is $E = 2P - 3V$, where E stands for external. The overall phase factor is $\mathrm{i}^{V-3V+P} = \mathrm{i}^{V+E-P}$. Choosing $3V$ vertices from $2P$ ends of propagators we have $(2P)!/(2P-3V)!$ diagrams. But many of them are identical. For one particular diagram, we can permute the vertices and get a factor $V!$, and permute the propagators and get a factor $P!$. We can also rearrange derivatives on a vertex which yields a 3! for each vertex and rearrange the ends of a propagator which yields a 2! for each propagator. The factors from the Taylot expansion is then neatly cancelled. We should also consider that by symmetry some of the permutations above is equivalent, for which we need a symmetry factor. For a general diagram D, we have n_I identical connected parts C_I. Sum over all the diagrams we have

$$Z_1(J) \propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} = \exp\left(\sum_I C_I\right). \tag{4.49}$$

Imposing the normalization condition $Z_1(0) = 0$ we can just omit all the vacuum diagrams, that is, diagrams without sources. Finally we have

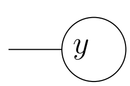
$$Z_1(J) = \exp\left(\mathrm{i}W_1(J)\right), \tag{4.50}$$

where

$$\mathrm{i}W_1(J) = \sum_{E\neq 0} C_I. \tag{4.51}$$

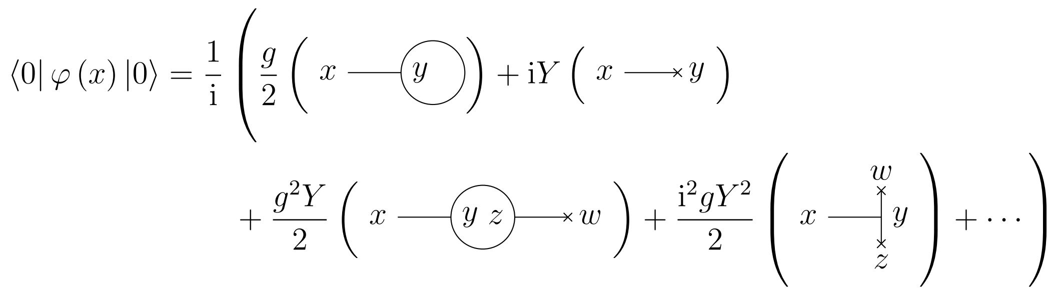
Were there no counterterms in $\mathcal{L}_1$ we would have $Z(J) = Z_1(J)$. Therefore

$$\begin{aligned} \langle 0|\varphi(x)|0\rangle &= \frac{\delta}{\mathrm{i}\delta J(x)} Z_1(J)\bigg|_{J=0} = \frac{\delta}{\delta J(x)} W_1(J)\bigg|_{J=0} \\ &= \frac{1}{\mathrm{i}}(\text{sum of } E=1 \text{ diagrams with source removed}) \\ &= \frac{g}{2\mathrm{i}}\ x \end{aligned}$$



$$\begin{aligned} &\quad + O\left(g^3\right) \\ &= \frac{g}{2\mathrm{i}} \int \mathrm{d}^4 y\, \Delta(x-y)\, \Delta(y-y) + O\left(g^3\right), \end{aligned} \tag{4.52}$$

which cannot be zero. Here we have $Z_g = 1$ for $Z_g = 1 + O\left(g^2\right)$. To fix this, introduce the counterterm $Y\varphi$, and we have



$$= \left(Y - \frac{\mathrm{i}g}{2}\Delta\left(0\right)\right)\int \mathrm{d}^4y\, \Delta\left(x-y\right) + O\left(g^3\right). \tag{4.53}$$

Then we should set

$$Y = \frac{\mathrm{i}g}{2}\Delta(0) + O\left(g^3\right) \tag{4.54}$$

to meet the normalization condition. We expect that Y be real and equivalently $\Delta\left(0\right)$ be pure imaginary. However, the integral

$$\Delta\left(0\right) = \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{1}{k^2+m^2-\mathrm{i}\epsilon}, \tag{4.55}$$

diverges. This is an example of ultraviolet divergence, and we introduce an ultraviolet cutoff Λ to deal with it by replacing the integral with

$$\Delta\left(x-y\right) \to \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{k^2+m^2-\mathrm{i}\epsilon}\left(\frac{\Lambda^2}{k^2+\Lambda^2-\mathrm{i}\epsilon}\right)^2. \tag{4.56}$$

For $\Lambda \gg m$, the result is

$$\Delta\left(0\right) = \frac{\mathrm{i}}{16\pi^2}\Lambda^2. \tag{4.57}$$

We can formally take the limit $\Lambda \to \infty$ so that Y becomes infinity leaving the $\langle 0|\,\varphi\left(x\right)|0\rangle$ zero. Note that Y is not physically observable. Once the requirement $\langle 0|\,\varphi\left(x\right)|0\rangle$ is met by the Y counterterm, except for the one source diagrams, all the diagrams which can be divided into one part without source and the other part with only one source by cutting a propagator are also cancelled.

Adding the remaining counterterms yields

$$Z\left(J\right) = \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\frac{\delta}{\mathrm{i}\delta J\left(x\right)}\left(-\left(Z_\varphi - 1\right)\partial_x^2 + \left(Z_m-1\right)m^2\right)\frac{\delta}{\mathrm{i}\delta J(x)}\right)Z_1\left(J\right). \tag{4.58}$$

This introduces to a new kind of vertex that two propogators join. The factor of the vertex is $(-\mathrm{i})\int \mathrm{d}^4x\,\left(-\left(Z_\varphi-1\right)\partial_x^2 + \left(Z_m-1\right)m^2\right)$.

5 Scattering amplitudes and the Feynman rules

6 Symmetries in quantum field theory

6.1 Noether's theorem

To evaluate the effect of a continuous global symmetry, we follow the way in book [2].

Consider an infinitesimal global symmetry transformation that transforms a field $\psi^l(x)$ to $\psi^l(x) + \mathrm{i}\epsilon\mathcal{F}^l(x)$ where $\mathcal{F}$ may generally depend on the field and its derivatives. By the notion 'symmetry' we mean that the action will not change for such a transformation (even though we may not apply the equation of motion). Now consider an arbitrary selection of $\epsilon(x)$ and a transformation

$$\psi^l(x) \to \psi^l(x) + \mathrm{i}\epsilon(x)\mathcal{F}^l(x). \tag{6.1}$$

Such a transformation induces a shift in action. Due to the requirement of symmetry the shift will vanish if $\epsilon(x)$ is a constant, so there exists some $J^\mu(x)$ called the conserved current such that

$$\delta S = -\int \mathrm{d}^4x\, J^\mu(x)\,\partial_\mu\epsilon(x). \tag{6.2}$$

Imposing the stationary condition for the action (which is equivalent to the equation of motion), we have for any infinitesimal transformation $\delta S = 0$. Integrating by parts and considering the arbitrariness of $\epsilon(x)$ then yields

$$\partial_\mu J^\mu(x) = 0. \tag{6.3}$$

And we immediately have for

$$F(t) = \int \mathrm{d}^3x\, J^0(t, \boldsymbol{x}), \tag{6.4}$$

the time derivative vanishes,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0. \tag{6.5}$$

We call F the conserved charge for it does not vary with time. This is the famous Noether's theorem: symmetries imply conservation laws.

If the requirement of symmetry is not only about the invariance of action but also about the invariance of Lagrangian, we can get the exact form of conserved charge. Consider an $\epsilon(x)$ depending only on time t, the shift of Lagrangian is therefore

$$\delta L = \mathrm{i}\int \mathrm{d}^3x\, \left(\frac{\delta L}{\delta\psi^l(t,\boldsymbol{x})}\epsilon(t)\mathcal{F}^l(t,\boldsymbol{x}) + \frac{\delta L}{\delta\dot{\psi}^l(t,\boldsymbol{x})}\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon(t)\mathcal{F}^l(t,\boldsymbol{x})\right)\right). \tag{6.6}$$

Due to the requirement that the Lagrangian is invariant under the global symmetry, that is, δL vanishes for constant $\epsilon(t)$, we have

$$\delta L = \mathrm{i}\int \mathrm{d}^3x\, \frac{\delta L}{\delta\dot{\psi}^l(t,\boldsymbol{x})}\dot{\epsilon}(t)\,\mathcal{F}^l(t,\boldsymbol{x})\,. \tag{6.7}$$

Comparing with equation (6.2), we have

$$F = -\mathrm{i}\int \mathrm{d}^3x\, \frac{\delta L}{\delta\dot{\psi}^l(t,\boldsymbol{x})}\mathcal{F}^l(t,\boldsymbol{x})\,. \tag{6.8}$$

If the requirement of symmetry is even stronger that the Lagrangian density be invariant, we can get the exact form of the conserved current. Similarly, for an arbitrary $\epsilon(x)$, the shift of Lagrangian density is

$$\delta\mathcal{L} = \mathrm{i}\left(\frac{\partial\mathcal{L}}{\partial\psi^l(x)}\epsilon(x)\,\mathcal{F}^l(x) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^l(x)\right)}\partial_\mu\left(\epsilon(x)\,\mathcal{F}^l(x)\right)\right). \tag{6.9}$$

Imposing the symmetry condition we have

$$\delta\mathcal{L} = \mathrm{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^l(x)\right)}\mathcal{F}^l(x)\,\partial_\mu\epsilon(x)\,. \tag{6.10}$$

Comparing with equation (6.2), we have

$$J^\mu(x) = -\mathrm{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^l(x)\right)}\mathcal{F}^l(x)\,. \tag{6.11}$$

In a quantum theory, the conserved charge is not only independent of time, but also a generator of the symmetry. Too see this, consider the symmetry of the Lagrangian. The symmetry transforms the canonical fields $Q^l(t,\boldsymbol{x})$ into $\boldsymbol{x}$-dependent functionals of themselves, that is, $\mathcal{F}^l(t,\boldsymbol{x}) = \mathcal{F}^l[Q(t),\boldsymbol{x}]$. The canonical conjugate of Q^l is $P^l = \delta L/\delta\dot{Q}^l$. Then we can rewrite equation (6.8) as

$$F = -\mathrm{i}\int \mathrm{d}^3x\, P_n(t,\boldsymbol{x})\,\mathcal{F}^n[Q(t);\boldsymbol{x}]\,. \tag{6.12}$$

Use the canonical quantization condition we have

$$[F, Q^n(t,\boldsymbol{x})] = -\mathcal{F}^n(t,\boldsymbol{x})\,. \tag{6.13}$$

As an example, consider the translational symmetry $\psi^l(x) \to \psi^l(x+\epsilon) = \psi^l(x) + \epsilon^\mu\partial_\mu\psi^l(x)$. That is, $\mathcal{F}^l_\mu = -\mathrm{i}\partial_\mu\psi^l$. We have four conserved currents grouped as the energy-momentum tensor $T^\mu{}_\nu$ such that $\partial_\mu T^\mu{}_\nu = 0$. The conserved charges are

$$P_\nu = \int \mathrm{d}^3x\, T^0{}_\nu\,. \tag{6.14}$$

Consider the Lagrangian density with a spacetime-dependent translation $\epsilon(x)$ we can write $T^{\mu}{}_{\nu}$ explicitly. Note that equation (6.11) cannot be applied here for the Lagrangian density is not invariant. The shift of the action is

$$\begin{aligned}\delta S[\psi] &= \int \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial \psi^l} \epsilon^\mu \partial_\mu \psi^l + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^l)} \partial_\nu \left(\epsilon^\mu \partial_\mu \psi^l \right) \right). \\ &= \int \mathrm{d}^4 x \left(\frac{\partial \mathcal{L}}{\partial x^\mu} \epsilon^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^l)} \partial_\mu \psi^l \partial_\nu \epsilon^\mu \right).\end{aligned} \tag{6.15}$$

Integrating the first term by part and compareing with equation (6.2) we have

$$T^{\nu}{}_{\mu} = \delta^\nu_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^l)} \partial_\mu \psi^l. \tag{6.16}$$

We further consider the quantum version of the conserved charge P_ν in equation (6.14). From equation (6.8) we know that

$$P_\mu = -\int \mathrm{d}^3 x \, P_n(t, \boldsymbol{x}) \, \partial_\mu Q^n(t, \boldsymbol{x}). \tag{6.17}$$

And hence the commutators

$$[P_\mu, \mathcal{O}(t, \boldsymbol{x})] = \mathrm{i} \partial_\mu \mathcal{O}(t, \boldsymbol{x}), \tag{6.18}$$

where $\mathcal{O}$ can be Q^n, P_n or any function of them.

6.2 Topological operators and symmetries

First we derive the Ward identity,

$$\partial_\mu \langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \mathrm{i} \sum_i \delta(x - x_i) \, \partial_i^\nu \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle. \tag{6.19}$$

As we have discussed, under a translational symmetry, the field will be changed by an infinitesimal translation as

$$\psi(x) \to \psi(x + \epsilon(x)) = \psi(x) + \delta_\epsilon \psi(x), \tag{6.20}$$

where $\delta_\epsilon \psi = \epsilon^\mu \partial_\mu \psi$. Since the action is invariant if ϵ is constant, there exists a tensor $T^{\mu\nu}$ such that

$$\delta_\epsilon S = -\int \mathrm{d}^d x \, \epsilon_\nu(x) \, \partial_\mu T^{\mu\nu}(x). \tag{6.21}$$

Consider the expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int \mathcal{D}\psi \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \, \mathrm{e}^{\mathrm{i}S[\psi]}. \tag{6.22}$$

By symmetry we expect that the infinitesimal translation leaves the path integral invariant,

$$\begin{aligned} 0 &= \int \mathcal{D}\psi\, \delta_\epsilon \left(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\, \mathrm{e}^{\mathrm{i}S[\psi]} \right) \\ &= \sum_i \langle \mathcal{O}_1(x_1) \cdots \delta_\epsilon \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle + \mathrm{i} \left\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \int \mathrm{d}^d x\, \epsilon_\nu(x)\, \partial_\mu T^{\mu\nu}(x) \right\rangle , \end{aligned} \tag{6.23}$$

where $\delta_\epsilon \mathcal{O}_i(x_i) = \epsilon_\nu(x_i)\, \partial_i^\nu \mathcal{O}_i(x_i)$. Take the functional derivative with respect to $\epsilon_\nu(x)$ we have the equation (6.19).

Consider the integral of $T^{\mu\nu}$ over a closed surface Σ,

$$P^\nu(\Sigma) = \int_\Sigma \mathrm{d}S_\mu\, T^{\mu\nu}(x) . \tag{6.24}$$

The correlator with other operators inside the surface

$$\langle P^\mu(\Sigma)\, \mathcal{O}(x) \cdots \rangle = \mathrm{i}\partial^\mu \langle \mathcal{O}(x) \cdots \rangle . \tag{6.25}$$

It indicates that the correlator is unchanged as we move the surface Σ without crossing any operator insertions. So we say that $P^\nu(\Sigma)$ is a topological surface operator.

We are going to show that in quantum field theory, having a topological codimensional-1 operator is the same as having a symmetry.

To specify a quantization, we foliate the spacetime into slices each associated with a Hilbert space of states $\mathcal{H}$. Chosen a quantization procedure, a correlation function $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ gets interpreted as a time-ordered expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \left\langle 0 \middle| \mathrm{T} \left\{ \widehat{\mathcal{O}}_1(t_1, \boldsymbol{x}_1) \cdots \widehat{\mathcal{O}}_n(t_n, \boldsymbol{x}_n) \right\} \middle| 0 \right\rangle , \tag{6.26}$$

where the time ordering is with respect to our foliation, $|0\rangle$ is the vacuum in the Hilbert space $\mathcal{H}$ and $\widehat{\mathcal{O}}_i : \mathcal{H} \mapsto \mathcal{H}$ is the quantum operators corresponding to the path integral insertions.

Let Σ_t be a spatial slice at time t, then $P^\mu(\Sigma_t)$ is invariant as we move the Σ_t for P^μ is topological. This is the path integral version of the conservation of momentum.

Consider two slices Σ_1, Σ_2 at t_1, t_2 respectively, and a insertion $\mathcal{O}(x)$ at time t, where $t_1 < t < t_2$. We have

$$\langle (P^\mu(\Sigma_2) - P^\mu(\Sigma_1))\, \mathcal{O}(x) \cdots \rangle = \left\langle 0 \middle| \mathrm{T} \left\{ \left[\widehat{P}^\mu, \widehat{O}(x) \right] \cdots \right\} \middle| 0 \right\rangle . \tag{6.27}$$

In the equation above we assume that other operators are outside the region closed by Σ_1 and Σ_2. Deform the two surfaces into a sphere surrounding the x, we have

$$\left[\widehat{P}^\mu, \widehat{\mathcal{O}}(x) \right] = \mathrm{i}\partial^\mu \widehat{\mathcal{O}}(x) \tag{6.28}$$

within the quantization or more generally

$$[P^{\mu}, \mathcal{O}(x)] = \mathrm{i}\partial^{\mu}\mathcal{O}(x)\,. \tag{6.29}$$

Note that although P^{μ} is nonlocal, the commutator with a local operator is local for the sphere surrounding x can be chosen arbitrarily small. Furthermore, integrate the equation above, we have

$$\mathcal{O}(x) = \mathrm{e}^{-\mathrm{i}Px}\mathcal{O}(0)\,\mathrm{e}^{\mathrm{i}Px}. \tag{6.30}$$

This statement is also true in any quantization of the theory.

We can now discuss the relation between the path integral formalism and the canonical quantization formalism. Equation (6.28), as a special case of equation (6.29), is exactly the same as equation (6.18), from either of which together with the definition of canonical conjugate and equation (6.17) (which is a direct result of the definition and the Noether's theorem) we can derive the canonical quantization conditions. The opposite path, deriving path integral formalism from canonical quantization is widely discussed in textbooks. Therefore we have shown that the path integral formalism and the canonical quantization formalism is equivalent.

7 Representations of Lorentz group

References

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